Exam I Review - Worksheet Math 1131 Class 16

Name: Solutions Section: ____________

Functions

1. Give the definition of the statement "$f(x)$ is a function".

each input has a single output

2. Let $f(x) = x^2 + x + 1$. Find $f(0)$, $f(1)$, and $f(2)$.

$f(0) = 0^2 + 0 + 1 = 1$

$f(1) = 1^2 + 1 + 1 = 3$

$f(2) = 2^2 + 2 + 1 = 4 + 2 + 1 = 7$

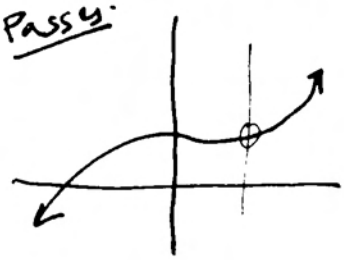
3. Let $f(x) = x^2 + x + 1$. Write down and simplify $f(x+1)$ and $\dfrac{f(1+h) - f(1)}{h}$.

$$f(x+1) = (x+1)^2 + (x+1) + 1$$
$$= x^2 + 2x + 1 + x + 1 + 1$$
$$= x^2 + 3x + 3$$

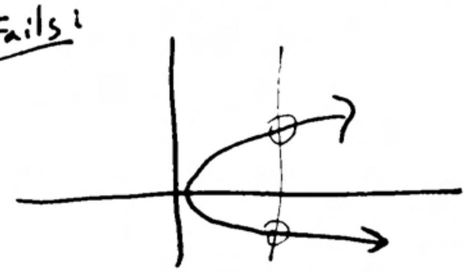
$$\frac{f(1+h) - f(1)}{h} = \frac{[(1+h)^2 + (1+h) + 1] - [1^2 + 1 + 1]}{h}$$
$$= \frac{1 + 2h + h^2 + 1 + h + 1 - 3}{h}$$
$$= \frac{3h + h^2}{h} = 3 + h$$

4. Sketch one graph that passes the vertical line test, and one that fails the vertical line test.

Passes:



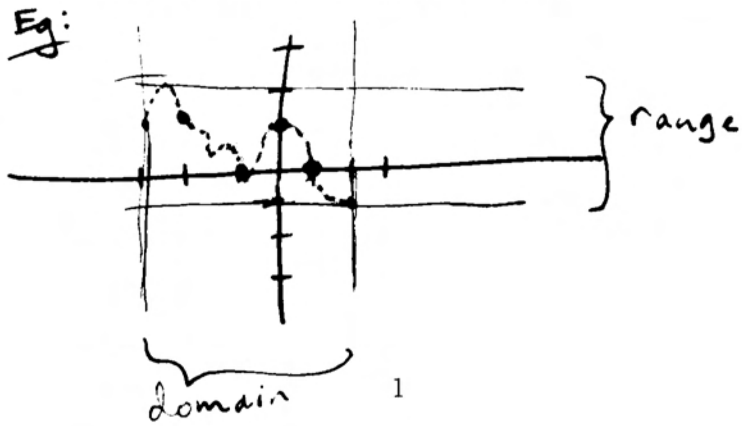
Fails:



5. Sketch the graph of a function with domain $[-3, 2]$ and range $[-1, 2]$ such that $f(-2) = 1$, $f(-1) = 0$, $f(0) = 1$, and $f(1) = 0$.

Your answers may vary a lot!

Eg:



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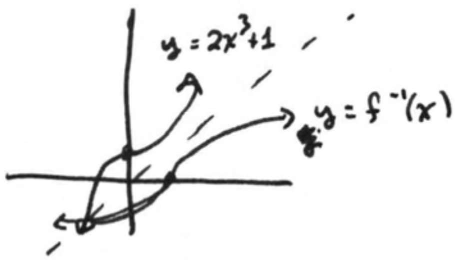
Inverses

1. Sketch a graph of $y = 2x^3 + 1$ and use this to sketch the graph of its inverse.



2. Let $f(x) = 2x^3 + 1$. What is $f^{-1}(3)$?

$f^{-1}(3) = x$

means $3 = f(x) = 2x^3 + 1$

so $2 = 2x^3$

$1 = x^3 \Rightarrow x = \sqrt[3]{1} = 1$

$f^{-1}(3) = 1$

$\boxed{\text{OR}}$ $f^{-1}(3) = x$

$3 = 2x^3 + 1$

Notice plugging in $x = 1$ gives $2 \cdot 1^3 + 1 = 3$

$\Rightarrow f^{-1}(3) = 1$

3. Let $f(x) = 2x^3 + 1$. Find an equation for $f^{-1}(x)$, and use it to compute $f^{-1}(3)$.

$f(x) = 2x^3 + 1$

solve for x: $y = 2x^3 + 1$

$y - 1 = 2x^3$

$\frac{y-1}{2} = x^3$

$x = \sqrt[3]{\frac{y-1}{2}} = f^{-1}(y)$

$\Rightarrow \boxed{f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}}$

$f^{-1}(3) = \sqrt[3]{\frac{3-1}{2}} = 1$

4. Let $f(x) = 2e^{x+1} + 3$. Find an equation for $f^{-1}(x)$.

solve for x: $y = 2e^{x+1} + 3$

$y - 3 = 2e^{x+1}$

$\frac{y-3}{2} = e^{x+1}$

$\ln\left(\frac{y-3}{2}\right) = \ln\left(e^{x+1}\right)$

$\ln\left(\frac{y-3}{2}\right) = x + 1$

$x = \ln\left(\frac{y-3}{2}\right) - 1 = f^{-1}(y)$

$\Rightarrow \boxed{f^{-1}(x) = \ln\left(\frac{x-3}{2}\right) - 1}$

5. Let $f(x) = \dfrac{x}{x+1}$. Find an equation for $f^{-1}(x)$.

solve for x: $y = \frac{x}{x+1}$

$y(x+1) = x$

$yx + y = x$

solve for x:

$y = x - yx$

$y = x(1-y)$

$\frac{y}{1-y} = x = f^{-1}(y)$

$\Rightarrow \boxed{f^{-1}(x) = \frac{x}{1-x}}$

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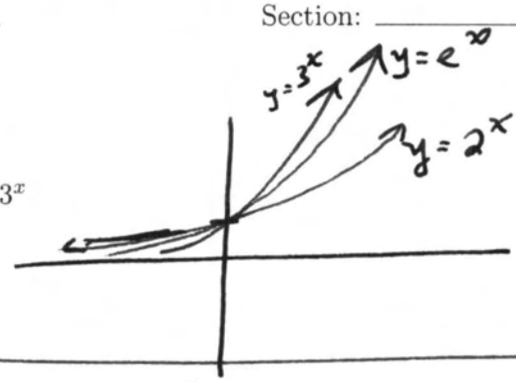
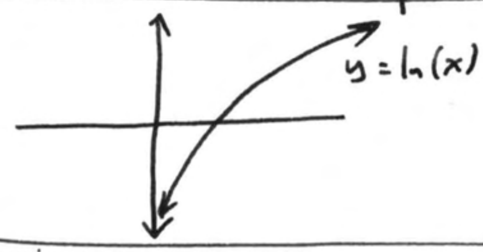
Exponentials and Logarithms

1. Sketch $y = e^x$, $y = 2^x$, and $y = 3^x$



2. Sketch $y = \ln(x)$



3. Simplify the expression:

$$\frac{(3y^2)^3}{y^4} = \frac{3^3 \cdot y^{2\cdot 3}}{y^4} = \frac{3^3 y^6}{y^4} = 3^3 y^2$$

4. Simplify the expression:

$$\frac{8^{-1/3}}{4^{-1/2}} = \frac{4^{\frac{1}{2}}}{8^{\frac{1}{3}}} = \frac{\sqrt{4}}{\sqrt[3]{8}} = \frac{2}{2} = 1$$

5. Simplify the expression:

$$\ln(x+1) + \ln(x-1) = \ln\left((x+1)(x-1)\right)$$
$$= \ln(x^2 - 1)$$

6. Simplify the expression:

$$3\ln(x) + 2\ln(x)$$
$$= \ln(x^3) + \ln(x^2)$$
$$= \ln(x^3 \cdot x^2) = \ln(x^5)$$

or

$$= 5 \cdot \ln(x)$$

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7. Solve for x:

$$e^{2x+1} = 5$$

$$\ln(e^{2x+1}) = \ln(5)$$

$$2x+1 = \ln(5)$$

$$2x = \ln(5) - 1$$

$$x = \frac{\ln(5) - 1}{2}$$

8. Solve for x:

$$\ln(2x+1) = 5$$

$$e^{\ln(2x+1)} = e^5$$

$$2x+1 = e^5$$

$$2x = e^5 - 1$$

$$x = \frac{e^5 - 1}{2}$$

9. Solve for x:

$$\frac{5e^{2x+1}}{5} = \frac{25}{5}$$

$$e^{2x+1} = 5$$

$$\ln(e^{2x+1}) = \ln(5)$$

$$2x+1 = \ln(5)$$

$$2x = \ln(5) - 1$$

$$x = \frac{\ln(5) - 1}{2}$$

10. Solve for x:

$$5e^{2x+1} = 2e^x$$

$$\frac{e^{2x+1}}{e^x} = \frac{\frac{2}{5}\cancel{e^x}}{\cancel{e^x}} \qquad \left(\frac{a^r}{a^s} = a^{r-s}\right)$$

$$e^{2x+1-x} = \frac{2}{5}$$

$$\ln\left(e^{x+1}\right) = \ln\left(\frac{2}{5}\right)$$

$$x+1 = \ln\left(\frac{2}{5}\right)$$

$$x = \ln\left(\frac{2}{5}\right) - 1$$

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$P(t) = P_0 \cdot e^{kt}$

Concrete Applications of Derivatives

1. Suppose that a population of bacteria is growing in a petri dish. Suppose also that the first time you look at the dish you count 20 bacteria, and that you count 200 bacteria in the dish 3 hours later. Find a formula for the population as a function of the number of hours t since your first measurement.

   How much time is required for the population to double in size?

$P(t) = 20 \cdot e^{kt}$

$\boxed{P(t) = 20 \cdot e^{\left(\frac{\ln(10)}{3}\right)t}}$

$P(3) = 200 = 20 \cdot e^{k \cdot 3}$

$10 = e^{k \cdot 3}$

$\ln(10) = k \cdot 3$

$k = \frac{\ln(10)}{3}$

find t s.t.

$40 = P(t)$

$40 = 20 \cdot e^{\frac{\ln(10)}{3}t}$

$2 = e^{\frac{\ln(10)}{3}t}$

$\ln(2) = \frac{\ln(10)}{3}t$

$\boxed{t = \frac{3 \cdot \ln(2)}{10}}$.

2. Suppose that Ebola is spreading through the city of Waterbury. Four people are ill two days into the outbreak, and eight people are ill four days in. Find a formula for the number ill a function of days since the outbreak began (0 days in).

   How long until 100 people are ill?

$P(\boxed{2}) = 4 = P_0 \cdot e^{k \cdot \boxed{2}}$

$P(\boxed{4}) = 8 = P_0 \cdot e^{k \cdot \boxed{4}}$

$P_0 = \frac{4}{e^{k \cdot 2}}$ so

$8 = \frac{4}{e^{k \cdot 2}} \cdot e^{k \cdot 4}$

$8 = 4 \cdot \frac{e^{4k}}{e^{2k}}$

$8 = 4 \cdot e^{4k - 2k}$

$8 = 4 \cdot e^{2k}$

$2 = e^{2k}$

$\ln(2) = 2k \Rightarrow k = \frac{\ln(2)}{2}$

$P(t) = P_0 \cdot e^{\frac{\ln(2)}{2}t}$

$P(2) = 4 = P_0 \cdot e^{\frac{\ln(2)}{2} \cdot 2}$

$4 = P_0 \cdot e^{\ln(2)}$

$4 = P_0 \cdot 2$

$P = 2$

$\boxed{P(t) = 2e^{\frac{\ln(2)}{2}t}}$

Solve for t

$100 = 2 \cdot e^{\frac{\ln(2)}{2}t}$

$\Rightarrow \dots$

$t = \frac{2 \cdot \ln(50)}{\ln(2)}$

3. Suppose that you begin with 100 grams of a radioactive substance. Suppose also that the substance has a half life of 3 years. Find a formula for the amount of radioactive substance remaining after t years.

   What is the weight of the radioactive substance that remains after 9 years?

$P(t) = 100 \cdot e^{kt}$

$\boxed{P(t) = 100 \cdot e^{\left(\frac{\ln(\frac{1}{2})}{3}t\right)}}$

$P(3) = 50 = 100e^{k \cdot 3}$

$\frac{1}{2} = e^{k \cdot 3}$

$\ln(\frac{1}{2}) = k \cdot 3$

$k = \frac{\ln(\frac{1}{2})}{3}$

$P(9) = 100 \cdot e^{\frac{\ln(\frac{1}{2})}{3} \cdot 9}$

$= 100 \cdot e^{\ln(\frac{1}{2}) \cdot 3}$

$= 100 \cdot \left(e^{\ln(\frac{1}{2})}\right)^3$

$= 100 \cdot \left(\frac{1}{2}\right)^3$

$= \frac{100}{8}$

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Limits and Continuity

1. Evaluate the following limit:

$$\lim_{x\to 3^-} \frac{x^2-3}{x+3}$$

(this is continuous at 3 $\Rightarrow$ we can plug in 3)

$$= \frac{(3)^2-3}{(3)+3} = \frac{9-3}{3+3} = \frac{6}{6} = 1$$

2. Evaluate the following limit:

$$\lim_{x\to 7} \frac{x^2+\sqrt{5x}-e^x}{x^2-7x+3}$$

(this is continuous at 7 $\Rightarrow$ we can plug in 7)

$$= \frac{7^2+\sqrt{5\cdot 7}-e^7}{7^2-7\cdot 7+3} = \frac{7^2+\sqrt{5\cdot 7}-e^7}{3}$$

3. Evaluate the following limit:

$$\lim_{x\to -3} \frac{x^2-9}{x+3}$$

($\underline{\text{NOT}}$ continuous at -3 $\Rightarrow$ must do more work)

$$= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}} = \lim_{x\to -3} (x-3)$$

$$= -6$$

4. Evaluate the following limit:

$$\lim_{x\to -3} \frac{x^2-9}{x^2+5x+6}$$

($\underline{\text{NOT}}$ continuous at -3 $\Rightarrow$ must do more work)

$$= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x+2)} = \lim_{x\to -3} \frac{(x-3)}{(x+2)}$$

$$= \frac{-6}{-1} = 6$$

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5. Evaluate the following limit:

cannot plug in 2

$$\lim_{x \to 2^-} \frac{x+1}{x-2}$$

$x \to 2$

think $\approx \dfrac{(\text{just under } 2)+1}{(\text{just under } 2)-2}$

$\approx \dfrac{\text{just under } 3}{\text{small neg } \#}$

$\approx$ Big neg #

$= -\infty$

6. Evaluate the following limit:

$$\lim_{x \to 2^+} \frac{x+1}{x-2}$$

$2 \leftarrow x$

Think $\approx \dfrac{(\text{just over } 2)+1}{(\text{just over } 2)-2}$

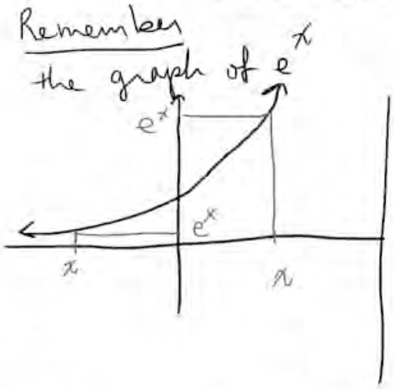
$\approx \dfrac{\text{just over } 3}{\text{small pos } \#}$

$\approx$ big pos #

$= \infty$

7. Evaluate the following limits:

Remember the graph of e^x



$$\lim_{x \to 0^-} e^{1/x}$$

$\approx e^{\frac{1}{\text{small neg}}}$

$\approx e^{\text{big neg}}$

≈ 0

$= 0$

and

$$\lim_{x \to 0^+} e^{1/x}$$

$\approx e^{1/(\text{small pos})}$

$\approx e^{\text{big pos}}$

$\approx$ BIG pos

$= \infty$

8. Evaluate the following limit:

$$\lim_{x \to 1^+} e^{-1/(1-x^2)}$$

$\approx e^{-1/\left(1-(\text{just over } 1)^2\right)}$

$\approx e^{-1/(\text{small neg})}$

$\approx e^{1/(\text{small pos})} \approx e^{\text{big pos}}$

$= \infty$

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9. Evaluate the following limit:

fastest growing

$$\lim_{x\to\infty}\frac{2x^3+3}{4x^2+7}$$

$$= \lim_{x\to\infty} \frac{x^3}{x^2}\cdot\frac{\left(2+\overbrace{\frac{3}{x^3}}^{\to 0}\right)}{\left(4+\underbrace{\frac{7}{x^2}}_{\to 0}\right)}$$

$$= \lim_{x\to\infty} x\cdot\frac{2}{4}$$

$$= \infty$$

10. Evaluate the following limit:

fastest growing

$$\lim_{x\to\infty}\frac{7x^2+4x}{3x^3+2x+7}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^3}\cdot\frac{\left(7+\overbrace{\frac{4}{x}}^{\to 0}\right)}{\left(3+\underbrace{\frac{2}{x}}_{\to 0}+\underbrace{\frac{7}{x^3}}_{\to 0}\right)}$$

$$= \lim_{x\to 0} \frac{1}{x}\cdot\frac{7}{3} = 0$$

11. Evaluate the following limit:

fastest growing

$$\lim_{x\to\infty}\frac{2x^3+3}{3x^3+7}$$

$$= \lim_{x\to\infty} \frac{x^3}{x^3}\cdot\left(\frac{2+\frac{3}{x^3}}{3+\frac{7}{x^3}}\right)$$

$$= \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}$$

12. Evaluate the following limit:

fastest growing

$$\lim_{x\to\infty}\frac{4x-2x^3-144+7x^2}{7-4x^2+2x-3x^3}$$

$$= \lim_{x\to\infty} \frac{x^3}{x^3}\cdot\frac{\left(\overbrace{\frac{4}{x^2}}^{\to 0}-2-\overbrace{\frac{144}{x^3}}^{\to 0}+\overbrace{\frac{7}{x}}^{\to 0}\right)}{\left(\underbrace{\frac{7}{x^3}}_{\to 0}-\underbrace{\frac{4}{x}}_{\to 0}+\underbrace{\frac{2}{x^2}}_{\to 0}-3\right)}$$

$$= \lim_{x\to\infty} 1\cdot\frac{(-2)}{(-3)}$$

$$= \frac{2}{3}$$

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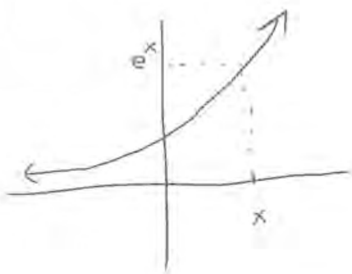
13. Evaluate the following limit:

fastest growing

$$\lim_{x\to\infty} \frac{4e^x - 2}{7 - 8e^x}$$

REMEMBER



$$= \lim_{x\to\infty} \frac{\cancel{e^x}}{\cancel{e^x}}\left(\frac{4 - \frac{2}{e^x}}{\frac{7}{e^x} - 8}\right)$$

$\frac{2}{e^x} \to 0$, $\frac{7}{e^x} \to 0$

$$= \frac{4}{-8} = \frac{-1}{2}$$

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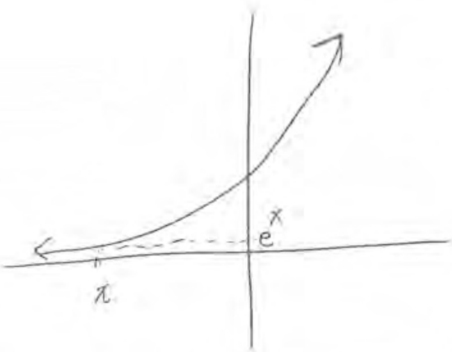
14. Evaluate the following limit:

$$\lim_{x\to-\infty} \frac{\overbrace{4e^x}^{\to 0} - 2}{7 - \underbrace{8e^x}_{\to 0}} = \frac{-2}{7}$$



NOTICE as $x \to -\infty$
e^x is
shrinking to 0,
NOT growing

15. Evaluate the following limit:

$$\lim_{x\to\infty} \frac{9\cos(4x + 1)}{x \cdot e^x}$$

two ways to do this

Think about it

when x is Big & Positive

$$\approx \frac{9(\#\text{ between } -1 \text{ \& } 1)}{\text{huge pos }\#}$$

$\approx$ tiny #

$= 0$

Squeeze Theorem

$$-1 \le \cos(4x + 1) \le 1$$

so

$$\underbrace{\frac{-9}{x \cdot e^x}}_{\to 0} \le \frac{9 \cdot \cos(4x+1)}{x \cdot e^x} \le \underbrace{\frac{9}{x \cdot e^x}}_{\to 0}$$

$$\Downarrow$$

$$0$$

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15. Find the vertical and horizontal asymptotes of the following function:

$$f(x) = e^{1/x}$$

only undefined at $x=0$

we've shown before $\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$

$\lim_{x\to 0^-} e^{\frac{1}{x}} = 0$

$\Rightarrow$ there is a vertical asymptote at $x=0$

to look for horizontal asymptotes,

$\lim_{x\to\infty} e^{\frac{1}{x}}$ $\left(\begin{array}{c}\text{as } x\to\infty \\ \frac{1}{x}\to 0\end{array}\right) = \lim_{\boxed{\frac{1}{x}}\to 0} e^{\boxed{\frac{1}{x}}} = e^0 = 1$

$\lim_{x\to -\infty} e^{\frac{1}{x}}$ $\left(\begin{array}{c}\text{as } x\to -\infty \\ \frac{1}{x}\to 0\end{array}\right) = \lim_{\boxed{\frac{1}{x}}\to 0} e^{\boxed{\frac{1}{x}}} = e^0 = 1$

$\Rightarrow$ there is a horizontal asymptote at $y=1$

16. Find the vertical and horizontal asymptotes of the following function:

$$f(x) = \frac{x+1}{x-2}$$

only undefined at $x=2$

& we've shown before

$\lim_{x\to 2^+} f(x) = \infty$

$\lim_{x\to 2^-} f(x) = -\infty$

$\Rightarrow$ there is a vertical asymptote at $x=2$

to find horizontal asymptotes

$\lim_{x\to\infty} \frac{x+1}{x-2} = \lim_{x\to\infty} \frac{\cancel{x}(1+\frac{1}{x}\overset{\to 0}{)}}{\cancel{x}(1-\underset{\to 0}{\frac{2}{x}})} = 1$

and $\lim_{x\to -\infty} \frac{x+1}{x-2} = \lim_{x\to -\infty} \frac{\cancel{x}(1+\overset{\to 0}{\frac{1}{x}})}{\cancel{x}(1-\underset{\to 0}{\frac{2}{x}})} = 1$

$\Rightarrow$ there is a horizontal asymptote at $y=1$

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Derivatives

Suppose $f(x)$ is some function.

1. Write down the limit definition of the derivative $f'(x)$.

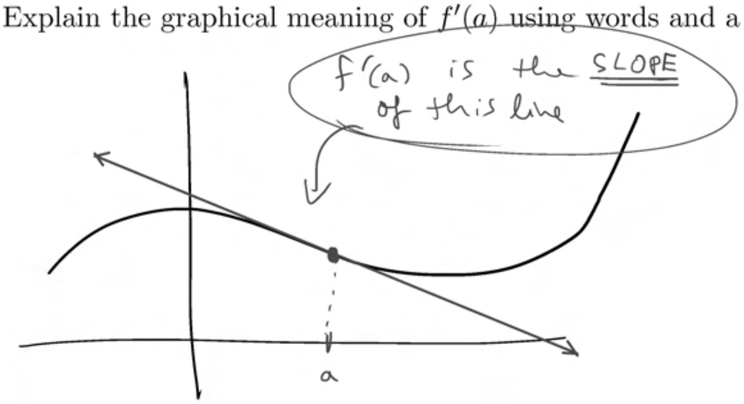
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Explain why the derivative $f'(a)$ gives the "instantaneous rate of change of f at a".

$$f'(a) = \lim_{h \to 0} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\text{average rate of change}}$$

over a smaller & smaller interval

3. Explain the graphical meaning of $f'(a)$ using words and a sketch.



$\boxed{f'(a) \text{ is the slope of } f(x) \text{ at } a}$

4. Write down an equation for the tangent line to $f(x)$ at the point $(a, f(a))$.

point – slope form: $y = m(x - x_1) + y_1$

TANGENT LINE $y = f'(a)(x - a) + f(a)$

Numbers (No x's)

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4. Compute the derivative of the following function:

$$\begin{array}{r} 32 \\ 4 \\ \hline 128 \end{array}$$

$$f(x) = 25x^5 + 32x^4 + 9x^3 + 144$$

$$f'(x) = \frac{d}{dx}\left(25x^5 + 32x^4 + 9x^3 + 144\right)$$

$$= \frac{d}{dx}\left(25x^5\right) + \frac{d}{dx}\left(32x^4\right) + \frac{d}{dx}\left(9x^3\right) + 144$$

$$= 125x^4 + 128x^3 + 27x^2 + 0$$

5. Compute the derivative of the following function:

$$f(x) = \sqrt{x} - 7e^x + (-1)\cdot x^9$$

$$f'(x) = \frac{d}{dx}\left[\sqrt{x} - 7e^x - x^9\right]$$

$$= \frac{d}{dx}\left[\sqrt{x}\right] - 7\cdot\frac{d}{dx}\left[e^x\right] - \frac{d}{dx}\left[x^9\right]$$

$$= \frac{1}{2}x^{-\frac{1}{2}} - 7e^x - 9x^8$$

6. Compute the derivative of the following function:

$$(fg)' = fg' + gf'$$

$$f(x) = (x^2 + x + 1)\cdot e^x$$

$$f'(x) = \frac{d}{dx}\left((x^2+x+1)e^x\right) = (x^2+x+1)\cdot\frac{d}{dx}\left[e^x\right] + e^x\cdot\frac{d}{dx}\left[x^2+x+1\right]$$

$$= (x^2+x+1)e^x + e^x(2x+1)$$

$$= e^x\left(x^2+3x+2\right)$$

7. Compute the derivative of the following function:

$$f(x) = e^x\cdot(\sqrt{x}+1) \qquad (fg)' = fg' + gf'$$

$$f'(x) = \frac{d}{dx}\left[e^x\cdot(x^{\frac{1}{2}}+1)\right] = e^x\cdot\frac{d}{dx}\left[x^{\frac{1}{2}}+1\right] + (x^{\frac{1}{2}}+1)\cdot\frac{d}{dx}\left[e^x\right]$$

$$= e^x\cdot\left[\frac{1}{2}x^{-\frac{1}{2}}\right] + (x^{\frac{1}{2}}+1)\cdot e^x$$

$$= \frac{e^x}{2\sqrt{x}} + e^x\sqrt{x} + e^x$$

8. For each of the above, write the equation for the tangent line to $f(x)$ at the point $(1, f(1))$.

for #4:

$$\left.\begin{array}{l} f(1) = 25 + 32 + 9 + 144 = 210 \\ f'(1) = 280 \end{array}\right] \Rightarrow y = 280(x-1) + 210$$

for #6:

$$\left.\begin{array}{l} f(1) = (1+1+1)e^1 = 3e \\ f'(1) = e^1(1^2+3\cdot1+2) = 6e \end{array}\right] \Rightarrow y = 6e(x-1) + 3e$$

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9. Compute the derivative of the following function:

$$\boxed{\left(\frac{t}{b}\right)' = \frac{b \cdot t' - t \cdot b'}{b^2}}$$

$$f(x) = \frac{e^x}{x}$$

$$f'(x) = \frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{x \cdot \frac{d}{dx}(e^x) - e^x \cdot \frac{d}{dx}(x)}{(x)^2}$$

$$= \frac{x \cdot e^x - e^x}{x^2} \quad \text{or} \quad = e^x\left(\frac{x-1}{x^2}\right)$$

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10. Compute the derivative of the following function:

$$\boxed{\left(\frac{t}{b}\right)' = \frac{b \cdot t' - t \cdot b'}{b^2}}$$

$$f(x) = \frac{x^2+1}{x^2+x+1}$$

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{x^2+x+1}\right) = \frac{(x^2+x+1) \cdot \frac{d}{dx}(x^2+1) - (x^2+1) \cdot \frac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{\overbrace{(x^2+x+1)(2x)}^{2x^3+2x^2+2x} - \overbrace{(x^2+1)(2x+1)}^{(2x^3+x^2+2x+1)}}{(x^2+x+1)^2} = \frac{\cancel{2x^3} + 2x^2 + 2x - (\cancel{2x^3} + x^2 + \dots}{(x^2+x+1)}$$

$$\boxed{f'(x) = \frac{x^2-1}{(x^2+x+1)}}$$

---

11. Compute the derivative of the following function:

$$\boxed{\left(\frac{t}{b}\right)' = \frac{b \cdot t' - t \cdot b'}{b^2}}$$

$$f(x) = \frac{x^2+1}{x^2-1}$$

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{(x^2-1) \cdot \frac{d}{dx}(x^2+1) - (x^2+1) \cdot \frac{d}{dx}(x^2-1)}{(x^2-1)^2} =$$

$$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{(2x^3-2x) - (2x^3+2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

---

12. For each of the above, write the equation for the tangent line to $f(x)$ at the point $(2, f(2))$.

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13. Compute the derivative of the following function:

constant coefficient

$$f(x) = 2\sin(x)\cdot\cos(x)$$

$$f'(x) = \frac{d}{dx}\left(\boxed{2}\sin(x)\cdot\cos(x)\right) = 2\cdot\frac{d}{dx}\left(\sin(x)\cdot\cos(x)\right)$$

$$(f\cdot g)' = f\cdot g' + g\cdot f'$$

$$= 2\cdot\left(\sin x\cdot\frac{d}{dx}(\cos(x)) + \cos(x)\cdot\frac{d}{dx}(\sin(x))\right)$$

$$= 2\left((\sin x)(-\sin(x)) + \cos(x)\cdot\cos(x)\right)$$

$$= 2\left(-\sin^2 x + \cos^2 x\right) = 2\cos^2 x - 2\sin^2 x$$

14. Compute the derivative of the following function:

$$\left(\frac{t}{b}\right)' = \frac{b\cdot t' - t b'}{b^2}$$

$$f(x) = \frac{\sin(x)}{x}$$

$$f'(x) = \frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{x\cdot\frac{d}{dx}(\sin(x)) - \sin(x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\cdot\cos(x) - \sin(x)}{x^2} \quad \text{OR} \quad = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}$$

Both are acceptable as final answers

15. Compute the derivative of the following function:

$$f(x) = \frac{\cos(x)+1}{\sin(x)-1}$$

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15. Compute the derivative of the following function:

$$f(x) = 5 \cdot (x^2 - 5)^{42}$$

$$f'(x) = \frac{d}{dx}\left[5 \cdot (x^2-5)^{42}\right]$$

$$= 5 \cdot 42 \cdot (x^2-5)^{41} \cdot \frac{d}{dx}[x^2-5]$$

$$= 210 \cdot (x^2-5)^{41} \cdot 2x$$

$$= 420\,(x^2-5)^{41}$$

16. Let $f(x) = \frac{3}{7-x}$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[3 \cdot (7-x)^{-1}\right]$$

$$= 3 \cdot (-1) \cdot (7-x)^{-2} \cdot \frac{d}{dx}[7-x]$$

$$= -3 \;\frac{1}{(7-x)^2} \cdot (-1)$$

$$f'(x) = \frac{3}{(7-x)^2}$$

17. Let $y = \sqrt{4x+2}$. Find y'.

$$y' = \frac{d}{dx}\left[(4x+2)^{\frac{1}{2}}\right]$$

$$= \frac{1}{2} \cdot (4x+2)^{-\frac{1}{2}} \cdot \frac{d}{dx}[4x+2]$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{4x+2}} \cdot 4$$

$$y' = \frac{2}{\sqrt{4x+2}}$$

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Higher Derivatives

1. Compute the second derivative of the following function:

$$f(x) = 25x^5 + 32x^4 + 9x^3 + 144$$

$$f'(x) = 125x^4 + 128x^3 + 27x^2$$

$$\boxed{f''(x) = 625x^3 + 384x^2 + 54x}$$

2. Let $f(x) = \sqrt{x} - 7e^x + (-1)\cdot x^9$. Find $f''(x)$.

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 7e^x - 9x^8$$

$$f''(x) = \frac{1}{2}\cdot\frac{-1}{2}\cdot x^{-\frac{3}{2}} - 7e^x - 72x^7$$

$$\boxed{f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} - 7e^x - 72x^7}$$

3. Compute the second derivative of the following function:

$$f(x) = x \cdot e^x$$

$$f'(x) = \frac{d}{dx}[x\cdot e^x] = x\cdot\frac{d}{dx}[e^x] + e^x\cdot\frac{d}{dx}[x]$$

$$= xe^x + e^x$$

$$f''(x) = \frac{d}{dx}[xe^x + e^x] = \frac{d}{dx}[xe^x] + \frac{d}{dx}[e^x] = \boxed{xe^x + 2e^x}$$

$$= (xe^x + e^x) + e^x$$

4. Let $y = \dfrac{e^x}{x}$. Find y''.

$$\left(\frac{t}{b}\right)' = \frac{bt' - tb'}{b^2}$$

$$y'(x) = \frac{d}{dx}\left[\frac{e^x}{x}\right] = \frac{x\cdot\frac{d}{dx}[e^x] - e^x\cdot\frac{d}{dx}[x]}{x^2}$$

$$= \frac{xe^x - e^x}{x^2}$$

because $\frac{d}{dx}[xe^x] = xe^x + e^x$

$$y'' = \frac{d}{dx}\left[\frac{xe^x - e^x}{x^2}\right] = \frac{x^2\cdot\left(\frac{d}{dx}[xe^x - e^x]\right) - (xe^x - e^x)\cdot\frac{d}{dx}[x^2]}{x^4} = \frac{x^2(xe^x + e^x - e^x) - 2x(xe^x-}{x^4}$$

$$= \frac{x^2(xe^x) - 2x(xe^x - e^x)}{x^4}$$

Exam I Review - Worksheet Math 1131 Class 16

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Tangents

1. Find the equation of the line tangent to $f(x) = 6x \cdot e^x$ at the point with $x = 0$.

Ⓐ equation of tangent is $y = m(x - x_1) + y_1$ $\quad (fg)' = f \cdot g' + g \cdot f'$

$x_1 = 0 \Rightarrow y_1 = f(0) = 6 \cdot 0 \cdot e^0 = 0$

so $y = m(x-0)+0 = mx$

Ⓑ equation for $f'(x) = \frac{d}{dx}[6x \cdot e^x] = 6x \cdot \frac{d}{dx}[e^x] + e^x \cdot \frac{d}{dx}[6x]$

$= 6xe^x + 6 \cdot e^x$

Ⓒ $m = f'(0) = 6 \cdot 0 \cdot e^0 + 6 \cdot e^0 = 6 \cdot 1 = 6$

$\Rightarrow \boxed{y = 6x}$ ← the tangent to $f(x)$ at 0.

2. Find the equation of the line tangent to $f(x) = \sec(x)$ at the point $(\frac{\pi}{4}, \sqrt{2})$

3. Find the equation of the line tangent to $f(x) = (2x+1)(5-x)$ at $a = 2$.

Ⓐ eqn. of tangent is $y = m(x-2) + f(2)$

where $f(2) = (2 \cdot 2 + 1)(5-2) = 5 \cdot 3 = 15$

Ⓑ $f'(x) = \frac{d}{dx}\left[(2x+1)(5-x)\right] = \frac{d}{dx}\left[10x - 2x^2 + 5 - x\right] = \frac{d}{dx}\left[9x - 2x^2 + 5\right]$

$f'(x) = 9 - 4x$

Ⓒ $m = f'(2) = 9 - 4 \cdot 2 = 1$

$\Rightarrow \boxed{y = 1 \cdot (x-2) + 15}$

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4. Let $F(x) = f(x) \cdot g(x)$. Suppose that $f(1) = 3$, $f'(1) = 4$, $g(1) = 2$ and $g'(1) = 3$.

Find the equation for the line tangent to $F(x)$ at $x = 1$.

tangent line

$\boxed{y = 17(x-1) + 6}$

line: $y = m(x - x_1) + y_1$

tangent line at 1: $x_1 = 1$

$y_1 = F(1) = f(1) \cdot g(1) = 3 \cdot 2 = 6$

$m = F'(1)$

$F'(x) = \frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

$F'(1) = f(1) \cdot g'(1) + g(1) \cdot f'(1) = 3 \cdot 3 + 2 \cdot 4 = 9 + 8 = 17$

5. Let $F(x) = (f(x))^2$. Suppose that $f(1) = 3$ and that $f'(1) = 2$.

Find the equation for the line tangent to $F(x)$ at $x = 1$. $y = m(x - x_1) + y_1$

$x_1 = 1$

$y_1 = F(1) = (f(1))^2 = 3^2 = 9$

$F'(x) = \frac{d}{dx}\left[(f(x))^2\right] = 2 \cdot (f(x))^1 \cdot f'(x)$

$m = F'(1) = 2 \cdot f(1) \cdot f'(1) = 2 \cdot 3 \cdot 2 = 12$

tangent line is

$\boxed{y = 12(x-1) + 9}$

6. Let $F(x) = \dfrac{1}{g(x)}$. Suppose that $g(1) = 2$ and $g'(1) = 3$.

Find the equation for the line tangent to $F(x)$ at $x = 1$.

tangent line

$\boxed{y = \frac{-3}{4}(x-1) + \frac{1}{2}}$

$y = m(x - x_1) + y_1$

$x_1 = 1$

$y_1 = F(1) = \frac{1}{g(1)} = \frac{1}{2}$

$F'(x) = \frac{d}{dx}\left[(g(x))^{-1}\right] = -1 \cdot (g(x))^{-2} \cdot g'(x) \Rightarrow m = F'(1) = \frac{-1}{(g(1))^2} \cdot g'(1)$

$= \frac{-1}{4} \cdot 3 = \frac{-3}{4}$

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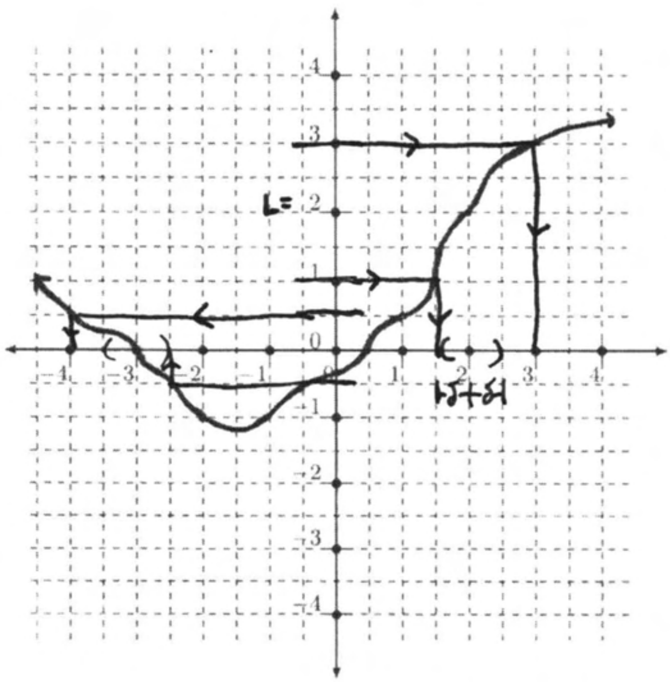
The Precise Definition of the Limit

Suppose that $f(x)$ is defined using the graph:



1. How close must x be to 2 to ensure that $f(x)$ is within 1 of 2? You must support your answer **by what you draw** in the figure.

we want f(x) within 1 unit of 2. [on the y-axis]

As long as x is within $\delta = .5$ of 2 [on the x-axis]

this will happen!

2. Find a value of δ such that if $0 < |x - (-3)| < \delta$, then $|f(x) - 0| < 0.5$. You must support your answer **by what you draw** in the figure.

This means " how ~~[illegible]~~ close must x be to $\boxed{-3}$ [on the x axis]

to ensure that f(x) is within $\boxed{0.5}$ of $\boxed{0}$?" [on the y-axis]

Answer: this happens when $\delta = 0.5$

i.e. when x is within 0.5 of −3

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3. Let $f(x) = 4x + 3$. How close must x be to 2 to ensure that $f(x)$ is within 0.1 units of 11 ? You must support your answer algebraically or graphically.

Want

$$|f(x) - 11| < 0.1$$

$$|4x + 3 - 11| < 0.1$$

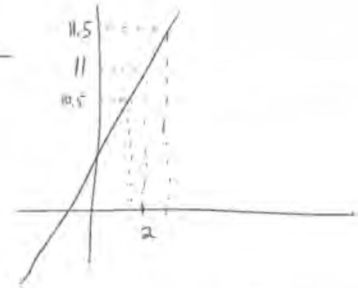
$$|4x - 8| < 0.1$$

$$|4(x-2)| < 0.1$$

$$|4| \cdot |x - 2| < 0.1$$

$$|x - 2| < \frac{0.1}{4} = \frac{1}{40} = 0.025$$

Idea



work backward from desired output error

Conclude

let $\delta = \frac{1}{40}$

If x is within $\frac{1}{40}$ of 2, then $f(x)$ is within 0.1 of 11

4. Let $f(x) = 2x - 1$. Find a value of δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.5$. You must support your answer algebraically or graphically.